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The recent successes of max-plus and more general idempotent structures for attacking nonlinear control problems offer the potential for revolutionary improvements in our ability to design and implement nonlinear controls for real applications. This effort has focused on theoretical foundations for the application of max-plus arithmetic in stochastic settings, as well as numerical methods of approximation.

In addition, a number of related issues in stochastic control have been considered. One problem of particular interest has been in the stability of estimation in stochastic adaptive control, and another has been in max-plus approaches for sensing, especially distributed sensing.

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Idempotent Methods for Continuous Time Nonlinear Stochastic Control Problems

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Abstract

The recent successes of max-plus and more general idempotent structures for attacking nonlinear control problems offer the potential for revolutionary improvements in our ability to design and implement nonlinear controls for real applications. This effort has focused on theoretical foundations for the application of max-plus arithmetic in stochastic settings, as well as numerical methods of approximation.

In addition, a number of related issues in stochastic control have been considered. One problem of particular interest has been in the stability of estimation in stochastic adaptive control, and another has been in max-plus approaches for sensing, especially distributed sensing.

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Objectives

In this effort, we have pursued a number of stochastic control problems. Our first objective has been in the application of max-plus methods in stochastic control. In particular, we have sought understanding of distributive idempotent techniques that provide efficient computational approaches in stochastic control problems.

In a second line of inquiry, we have also studied some stochastic adaptive control problems, specifically problems of adaptive disturbance cancellation. These problems arise in beam control for high-energy laser applications. We have maintained close collaboration with AFRL and Boeing Direct Energy Systems scientists and engineers in studying these applications.

Background

The max-plus algebra involves a redefinition of arithmetic operations, for computational and analytical benefit. The basic scalar set of interest is the real numbers, augmented by $-\infty$: $\mathbf{R}^- = \mathbf{R} \cup \{-\infty\}$. On this set, two operations, $\oplus$ and $\otimes$, are defined by

$$a \oplus b = \max\{a,b\},$$

$$a \otimes b = a + b.$$

It is well known that $\mathbf{R}^-$ forms a commutative semi-ring under these operations. The additive identity is $-\infty$ while the multiplicative identity is 0. Except for the additive identity, every element has a multiplicative inverse, suggesting that one might be able to extend the structure to a field structure. However, addition in this semi-ring is idempotent, meaning that $a \oplus a = a$. It is important to note that the only rings satisfying additive idempotency are trivial; that is, the only element is the additive identity. Thus, extending the semi-ring to a ring (and hence a field) is not a possibility.

From these basic operations, standard linear algebraic objects can be built, such as matrices and vectors.

To illustrate the application of max-plus algebraic structure, we consider a standard nonlinear control problem. We begin with a dynamical system under control, of the form

$$\dot{x} = f(x,u), \quad x(t_0) = x_0$$

with a control objective given by

$$J(u, x_0, t_0) = \int_{t_0}^{t_f} g(x(t), u(t)) dt,$$

which is to be maximized of the set of admissible control functions, $u \in U(t_0, t_f) \subset L^2(t_0, t_f)$. For a given control function and initial state, we denote the solution of the differential equation by $x(\bullet; t_0, x_0, u)$. The Bellman equation of dynamic programming, given by

$$V(y,t) = \max\left\{\int_t^s g(x(\tau), u(\tau)) d\tau + V(x(s;t, y, u), s)\right\}$$

leads to a family of operators $S_{s,t}$,

$$S_{s,t}(\phi) = \max\left\{\int_t^s g(x(\tau), u(\tau))d\tau + \phi(x(s;t,y,u))\right\}.$$

$$= \bigoplus_u \{G(u) \otimes L_u(\phi)\}$$

which are max-plus linear evolution operators.

The dynamic programming propagation operator in the stochastic case,

$$S_{s,t}(\phi) = \max\left\{E\left[\int_t^s g(X(\tau), u(\tau))d\tau + \phi(X(s;t,y,u))\right]\right\},$$

however, is not max-plus linear. However, we may use the distributive property of multiplication over addition, to apply max-plus effectively.

The max-plus distributivity principle we have developed is of the form

$$\int_W \sup_{z \in Z} h(w,z) P(dw) = \sup_{Q \in \mathbf{P}(W \times Z)} \int_W h(w,z) dQ(dwdz),$$

in which $\mathbf{P}(W,Z)$ denotes the set of probability measures on $W \times Z$ having marginal P on W.

The relevance of this result arises in conjunction with max-plus finite element approximations to the Bellman equations.

Max-Plus Finite Element Approximations

Max-plus finite elements provide useful approximation tools for dynamic programming. Within the context of this work, we use the finite elements in conjunction with max-plus distributivity to approximate solutions to the stochastic Bellman dynamic programming equation. We have considered the linear elements

$$\psi_i(x) = -c_i|x - x_i|,$$

quadratic elements

$$\psi_i(x) = -c_i|x - x_i|^2,$$

and Legendre elements

$$\psi_i(x) = p_i^T x - c|x|^2.$$

In the linear and quadratic elements, x_i are the element nodes, and c_i are scale parameters. In the Legendre formulation, the elements A max-plus approximation of a function f takes the form

$$f(x) \approx \bigoplus_{k=1}^N a_k \otimes \psi_k(x) = \max_k\{a_k + \psi_k(x)\}$$

in which the weights a_i are defined by

$$a_i = -\max_x\{\psi_i(x) - f(x)\}.$$

Note that a max-plus interpolation has an interesting and perhaps unintuitive structure. Figure 1 below illustrates the projection onto linear elements for an example function.
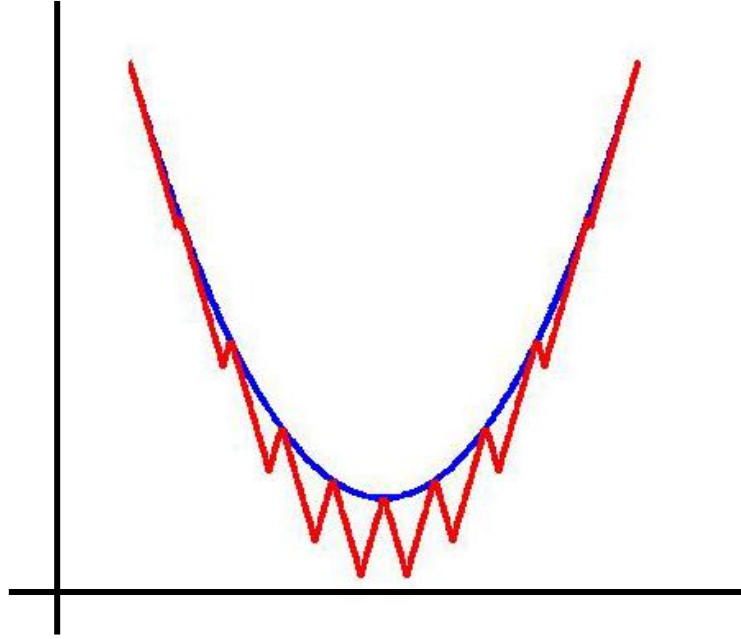
Figure 1. Blue curve is the original function; red curve is the max-plus finite element projection.

To apply the finite element method to dynamic programming, we first examine the deterministic Bellman equation

$$V(y,t) = \max\left\{\int_t^s g(x(\tau),u(\tau))d\tau + V(x(s;t,y,u),s)\right\}$$

which is max-plus linear. We plug in the finite element expansion

$$V^N(x,t) = \bigoplus_{k=1}^{N} a_k(t) \otimes \psi_k(x) = \max_k\{a_k(t) + \psi_k(x)\}$$

into the Bellman equation, which leads to the max-plus matrix iteration

$$a(t) = B \otimes a(t+h),$$

in which the matrix B is defined by

$$B_{ij} = -\max_x\left\{\psi_i(x) - S_{t,t+h}(\psi_j)(x)\right\}.$$

Our contribution on the deterministic side has been the introduction of the Legendre elements, which provide second order accuracy of approximation. Our focus, however, has been on the stochastic control area.

Traditional Stochastic Control Problems

The standard stochastic control model is of the form

$$dX = f(X,u)dt + \sigma(X)dW, \quad X(t_0) = X_0$$

in which W is a standard Brownian motion, and the standard running cost criterion is given by

$$J(u,x_0,t_0) = E\left[\int_{t_0}^{t_f} g(X(t),u(t))dt\right]$$

which is to be maximized over admissible controls

$$u \in U(t_0, t_f) \subset \left\{ u : [t_0, t_f] \to \mathbf{R}^m \mid E \int_{t_0}^{t_f} |u|^2 < \infty, u \text{ progressively measureable} \right\}.$$

The Bellman equation of dynamic programming (DPE) takes the form

$$V(y,t) = \max_{u \in U(t,s)} \left\{ E \left[\int_t^s g(X(\tau), u(\tau)) d\tau + V(X(s;t, y, u), s) \right] \right\}$$

in which V is the value function

$$V(x_0, t_0) = \max_{u \in U(t_0, t_f)} (J(u, x_0, t_0)).$$

The semigroup for backward propagation,

$$S_{s,t}(\phi) = \max \left\{ E \left[\int_t^s g(X(\tau), u(\tau) d\tau + \phi(X(s;t, y, u)) \right] \right\},$$

is not (necessarily) linear, because the maximization and expectation cannot be interchanged in order. However, max-plus distributivity allows us some flexibility.

The finite element expansion plugged into the semigroup yields

$$S_{s,t}(V^N) = \max \left\{ E \left[\int_t^s g(X(\tau), u(\tau)) d\tau + \bigoplus_{i=1}^N a_i(s) \otimes \psi_i(X(s;t, y, u)) \right] \right\}$$

$$= \bigoplus_u \left\{ \int_\Omega \bigoplus_{i=1}^N \left(\int_t^s g(X(\tau), u(\tau) d\tau \otimes a_i(s) \otimes \psi_i(X(s;t, y, u)) \right) \right\}.$$

Applying distributivity, we have

$$S_{s,t}(V^N)(y) = \bigoplus_{u \in U(s,t)} \bigoplus_{Q \in \mathbf{P}(\Omega \times \mathbf{N})} \left\{ \int_\Omega \int_s^t g(X_\tau, \mu_\tau) d\tau \otimes \int_\Omega a_z(t) \otimes \psi_z(X(t;s, y, u)) Q(\omega, z) \right\}$$

in which the distributivity property of the theorem involves the set of random variables taking values in the set $\mathbf{N} = \{1, 2, \ldots, N\}$ for the interchange of expectation and maximization order. This propagation is then projected back onto the finite element basis through the relation as in the deterministic situation. It is interesting to note that the optimization over Q is essentially a linear programming problem.

An alternative to applying distributivity involves the fast Legendre transform. Given a discretized function on a grid in $\mathbf{R}$, $\{(x_i, f_i) : 0 \le i \le n\}$, and discrete slope parameters $\{p_i : 0 \le i \le m\}$, we compute the approximate gradients

(x_j, g_j), $g_j = (y_{j+1} - y_j)/(x_{j+1} - x_j)$, $j = 0, \cdots, n-1$

and merge the two sequences g and p. The interval (p_i, p_{i+1}) that contains g_j provides the output slope of the discrete Legendre transform. If the sequences are pre-sorted, this merge takes $O(n + m)$ operations. Similar algorithms exist for higher dimensional problems.

Thus the fully discrete Legendre finite element algorithm consists of the following steps.
1. Identify the preconditioning quadratic function that makes the value convex.
2. Initialize at the final time T with value $V=0$ (or a non-zero exit time cost).
3. Back propagate the value function on a discrete grid of points to the next earlier time.

4. Re-project the value function onto the Legendre basis using the fast discrete Legendre transform.
5. Return to 3 and repeat until the initial time is reached.

Stochastic Adaptive Control Problems

Another focus of this research project has been in the analysis of adaptive control problems that arise in pointing and tracking for high-energy lasers. Generally speaking, these systems comprise a large, complex set of mechanical and electronic components. The primary goal in tracking is to maintain the target of interest in the center of the focal plane of a tracking camera. At the coarse physical scale of operation, this goal is attacked with a gimbal that rotates the telescope of the optical pointing and tracking system. At the fine scale, actuation is achieved with a fine track mirror that is controlled at a much higher frequency. In the material below, we discuss the fine track problem.

The fine track system is typically characterized with a linear time invariant model, instantiated as a discrete time rational transfer function. Of course the actual plant is the hardware: the model provides a means of devising control actions. A block diagram, denoting the operation of the fine track loop is given in Figure 2. The process involves an actual plant (that is, the hardware system to be controlled), a model of the plant, the controller, and the adaptive processing that tunes the gains of the controller to cancel the disturbance.
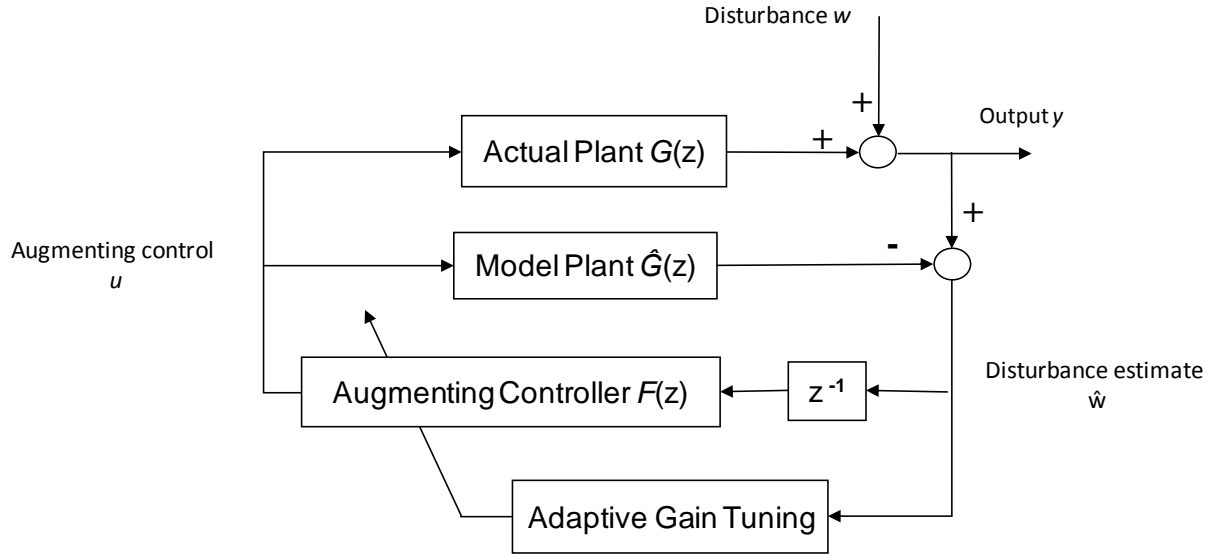

Figure 2. Adaptive control block diagram.

The actual plant is unknown and instantiated in hardware. The model plant is a mathematical and computational approximation of the actual plant that must be estimated experimentally. The augmenting controller attempts to produce a control signal that will cancel out the disturbance, w, and drive the output y to 0, using adaptive tuning of the tap weights or gains in an FIR filter. The model, in equation form, is given by

$$y = Gu + w = \text{actual hardware system}$$
$$\hat{w} = y - \hat{G}u = \text{estimate of disturbance}$$
$$u = -z^{-1}F\hat{w} = \text{adaptive controller}$$

The model of the plant provides a means of estimating the disturbance and determining the appropriate control with which to cancel it out. The control gains embodied in the filter F are determined by minimizing

$$J(F) = E\|y\|^2 = E\|(I + z^{-1}\hat{G}F)\hat{w}\|^2,$$

in which

$$y = \hat{w} + \hat{G}u = (I + z^{-1}\hat{G}F)\hat{w}$$

yields an estimate of the closed-loop noise-to-track error transfer function. Note that if we were to have estimated the transfer function perfectly, ($G = \hat{G}$), then this relationship would provide exactly the noise-to-output transfer function. We should also note that if $F = -z\hat{G}^{-1}$ were realizable, then this feedback operator would exactly cancel the transfer function. Since such an F is not causal (requiring at least one step of future data), the minimizing solution must depend on the disturbance. We use recursive least squares techniques to estimate the filter F.

With George Yin and Le Yi Wang of Wayne State, we have developed conditions under which least squares estimators converge, even under model/plant mismatch. These preliminary results are among the first approaches to analyzing the stability of these adaptive loops, a key issue to engineers building high energy laser systems.

The basic problem can be written as

$$y_n = \phi_n^T \theta + \Delta(\phi_n, \theta) + \varepsilon_n$$

in which Δ denotes the model mismatch, which may depend on the "true" parameter as well as the exogenous sequence ϕ. Applying a traditional adaptive estimation algorithm of the form

$$\theta_{n+1} = \theta_n + a_n\phi_n(y_n - \phi_n^T\theta_n)$$

leads to a biased estimator whose convergence properties are uncertain. Using weak convergence techniques, we have shown that sufficiently small plant perturbations lead to convergence of the adaptive estimation technique.

We should note that this technology has been involved in a number of transition efforts. In fact the actual application of the recursive estimation algorithms has preceded the theoretical understanding of stability: these algorithms have been applied by our team at White Sands Missile Range and Kirtland Air Force Base in laboratory field experiments, prototype strategic relay systems, and prototype tactical laser tracking platforms. Our technical point of contact at AFRL has been Dr. Dan Herrick, and we have worked closely with Steve Baugh of Boeing Directed Energy Systems in transition and testing as well.

Publications and Presentations

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7. "Max-Plus Methods for Computational Stochastic Control," Invited Presentation, AFOSR Workshop on Computational Issues in Nonlinear Control, Monterey, C A, Nov 9-10, 2009
8. "Control of Agent Models through Reduced Order Approximations," Invited Presentation, NIMBioS Investigative Workshop on Optimal Control of Individual and Agent Based Simulations, Knoxville, TN, Dec 1-3 2009.
9. "Max Plus Algebra and Computation of Nonlinear Controls," Colloquium talk, Mathematics Department, University of Louisiana at Lafayette, February 18, 2010.
10. "Legendre Finite Element Methods and Max-Plus Control," Invited Presentation, SIAM Conference on Control Theory and Its Applications, Baltimore, July 25-27 2011.
11. "Nonlinear Tracking and Max-Plus Control," Invited lecture, Computational Issues in Nonlinear Control, Monterey, CA, Nov 7-8, 2011.